Revision Exercise 4:
Differentiation & Applications

(1). A particle travels in a straight line, starting from rest at point A, passing through point B and coming to rest again at point C. The particle takes 5s to travel from A to B with constant acceleration. The motion of the particle from B to C is such that its speed, v ms^{-1}, t seconds after leaving A, is given by

$$v = \frac{1}{225}(20 - t)^3 \ for \ 5 \le t \le T.$$

(i) Find the speed of the particle at B and the value of T.
(ii) Find the acceleration of the particle when t = 14.
(iii) Sketch the velocity-time curve for $0 \le t \le T$.
(iv) Calculate the distance AC. (J02/P2/Q12)

(2) A particle moves in a straight line so that, t seconds after leaving a fixed point O, its displacement, s metres from O, is given by s $= 9t^2 + 6t^3 - 2t^4$. Find
(i) the positive value of t for which the particle is instantaneously at rest,
(ii) the total distance travelled by the particle from t = 0 to t = 4,
(iii) the acceleration of the particle when t = 1. (D00/P1/Q7)

(3) A particle moves in a straight line so that, at time t seconds after leaving a fixed point O, its velocity, $v\ m\ s^{-1}$, is given by v = $20\ e^{-\frac{t}{4}}$.
(i) Sketch the velocity-time curve.
(ii) Find the value of t when v = 10.
(iii) Find the acceleration of the particle when v = 10.
(iv) Obtain an expression, in terms of t, for the displacement from O of the particle at time t seconds. (D98/P2/Q5)

(4) A particle moves in a straight line so that at time t seconds after passing through a fixed point O, its displacement s m is given by $s = 2t(t - 3)^2$. Find

(i) the times when the particle is momentarily at rest,
(ii) the total distance traveled in the first 5 seconds,
(iii) the interval of time during which the velocity is negative and sketch the velocity-time graph for the first 5 seconds.

(5) A particle travels in a straight line so that at time t seconds, its distance s metres from the origin O is given by $s = t^3 - 6t^2 + 15t$.

(a) Show that the velocity is always positive.
(b) Find the least velocity attained.
(c) Find the range of values of t for which the particle is decelerating.
(d) Find the distance of the particle from O when the acceleration of the particle is instantaneously zero.
(e) Find the average velocity of the particle during the first 2 seconds.

(6) The diagram below shows a vertical cross-swection of a container in the form of an inverted cone of height 60cm and base radius 20cm. The circular base is held horizontal and uppermost. Water is poured into the container at a constant rate of 40
(i) Show that, when the depth of water in the container is x cm, the volume of water in the container is $\frac{\pi x^3}{27}\ cm^3$.
(ii) Find the rate of increase of x at the instant when x = 2. (D98/P1/Q6)

(7) The diagram shows a quadrilateral ABCD in which $\angle BCD$ *and* $\angle DAB$ are right angles, AB = 8 cm, BD = 10cm and $\angle BDC = \theta$ *where* $0^o < \theta < 90^o$.

(i) Show that the area S cm^2 of the quadrilateral ABCD is given by

$$S = 24 + 50\sin\theta\cos\theta$$

(ii) Hence find the maximum value of S and the value of θ when S is a maximum.

(8a) The tangent to the curve y = $px^2 + qx + 2$ *at* $\left(1, \frac{1}{2}\right)$ is parallel to the normal to the curve y = $x^2 + 6x + 4$ *at* $(-2, -4)$. *Find the value of p and of q*.

(b) An open rectangular fish tank of capacity 1152 cm^3 is to be constructed using materials of negligible thickness. If the length of the fish tank is 3x cm while its width is x cm, show that the amount of material needed to build the tank is given by A cm^2 where A = $3x^2 + \frac{3072}{x}$. Find the value of x for which A is a minimum.

(9) In the figure, PQRS is a square plastic plate of side 4 cm and ABCD is a square whose centre coincides with that of PQRS. The shaded regions are to be cut off an the remaining plastic is folded to form a right pyramid with base ABCD. Let AB = 2x cm and let V be the volume of the pyramid.

(i) Show that the height of the pyramid is $2\sqrt{1-x}$.

(ii) Show that V = $\frac{8}{3}x^2\sqrt{1-x}$.

(iii) Find the value of x such that V is maximum.

(10i). Find the equation of the normal to the curve $y = \frac{2x+4}{x-1}$ at the point where the curve meets the x-axis.

(ii) Given that $y = Ae^{kx}$, where A and k are constants, find an expression for $\frac{dy}{dx}$. Hence find the value of k and of A for which $\frac{dy}{dx} - 3y = 4e^{2x}$. (D01/P2/Q3)

(11) The gradient of a curve at any point is given by $\frac{dy}{dx} = 2 - \frac{x^3}{8}\cdot$ The curve intersects the x-axis at the point P. Given that the gradient of the curve at P is 1, find the equation of the curve.

(12) A curve has the equation y = $2\tan^2 x - 7\tan x,$ Find

(i) an expression for the gradient of the curve,

(ii) the x-coordinate of each of the stationary points of the curve for which $0 < x < 2\pi$ radians. (J00/P2/Q6)

(13) Given that y = $7 - 5x + 6x^2 - 3x^3,$

(a) find $\frac{dy}{dx},$

(b) find the value of k for which x + y = k is a tangent to the curve,

(c) show that y decreases as x increases.

(14) Two variables, x and y, are related by the equation $y = \frac{3}{4}\left(\frac{x}{12} - 1\right)^6 \cdot$ Given that both x and y vary with time, find the value of y when the rate of change of y is 12 times the rate of change of x. (D01/P1/Q16)

(15) Variables p and q are connected by the equation $pq^2 = 144\cdot$ Find an expression, in terms of q, for $\frac{dp}{dq}$ and hence find the approximate change in p as q increases from 6 to 6 + k, where k is small. (J98/P1/Q2)

(16) The variable y is given in terms of x by y = $\cos^2 x - 0.5,\ where\ 0 \le x \le \frac{\pi}{2}\cdot$ Given that x is increasing at 0.5 radians per second, find the rate of change of y with respect to time when x = $\frac{\pi}{6}\cdot$ (J96/P2/Q6ac)

(17) The time T taken by a planet to revolve around the sun and its mean distance r from the sun are related by T = k $r^{\frac{3}{2}}$ where k is a constant. Obtain $\frac{dT}{dr}\cdot$

If the planet's mean distance from the sun was to be increase by 2%, estimate the approximate percentage increase in the period T.

(18) A vessel is in the shape of an inverted right circular cone whose base-radius is equal to its height and whose axis is vertical. Liquid is poured into the vessel at a constant rate of $100\, cm^3 s^{-1}$. The volume of liquid in the vessel is $\frac{1}{3}\pi x^3\ cm^3$ when the depth of liquid is x cm. Calculate, at the instant when the depth of liquid is 10 cm, the rate of increase of

(i) the depth of the liquid,

(ii) the area of the horizontal surface of the liquid.

(b) Given that y = $x^3 + 3x^2$ use calculus to find, in terms of p, the approximate percentage increase in y when x increases from 2 by p%, where p is small. (D95/P1/Q13)

(19) The diagram below shows a circle of radius r cm inside an equilateral triangle of side x cm. It is given that x has an initial value of 10 cm and r has an initial value of 3 cm. Both x and r are increasing at the rate of 0.2 cm/s.

(i) Express x in terms of r and show that the shaded area, A cm^2, is given by

$$A = \frac{\sqrt{3}}{4}(r+7)^2 - \pi r^2.$$

(ii) Hence, find the rate of change of the shaded area after 20 seconds.

(20) In the diagram,PQ is a straight line. A is on PQ and triangle APB is right-angled at P. BP = 10 cm and PA = x cm. AB is the diameter of the semi-circle. Find

(i) Find in terms of x, the area A, of the semi-circle,

(ii) Given that A moves along PQ such that x is increasing at 0.8 cm per second, find the rate of increase in the area of the semi-circle at the instant when x = 12 cm.

(21) A container is such that when the depth of liquid in it is x cm, the volume is V cm^3, where $V = \frac{\sqrt{3\pi x^5}}{5}$.

(a) A small increase ∂x in the depth of the liquid in the container leads to a small increase ∂V *in its volume*. The depth is initially 10 cm, and then the volume is increased by 10 cm^3. Calculate the approximate increase in the depth.

(b) Show that the percentage increase in V is always approximately 2.5 times the percentage increase in x.

(c) Given that the volume is increasing at a rate of 20 cm^3 per second. Calculate the rate at which the depth of the liquid in the container is increasing when x = 15.

(22) The curve whose equation is y = $\left(2x^2 + 3x - 9\right)(x - k)$, where k is a constant, has a turning point where x = -1.

(i) Calculate the value of k.

(ii) Calculate the value of x at the other turning point on the curve.

(iii) Draw a rough sketch of the curve and find the set of values of x for which y > 0.

(D00/P2/Q1a)

(23) For the curve y = $\left(x^2 - 4\right)(2x - 1)$, calculate the coordinates of

(i) the points of intersection with the axes,

(ii) the turning points.

Hence sketch the curve. (D96/P2/Q1c)

(24) The variables θ and t are related by the equation $\theta = \theta_o e^{-kt}$, where θ_o and k are constants. When t = 30, $\theta = \frac{1}{2}\theta_o$.

(i) Show that the value of k, correct to 4 decimal places, is 0.0231.

When t = 40, $\theta = 28$.

(ii) Calculate the value of θ_o.

When t = 50, calculate

(iii) θ,

(iv) $\frac{d\theta}{dt}$.

Find the average rate of change of θ with respect to t over the interval $0 \le t \le 50$.

(D00/P2/Q3b)

(25) Differentiate the following expressions with respect to x.

(i) $\frac{2x - 1}{x + 5}$

(ii) $\sqrt{25 - 5x^3}$ (D96/P2/Q5ac)